Variational Principles for Compressible and Incompressible Systems

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Variational principles are established in the frame of macroscopic fluid theory including anisotropic pressure to compare an incompressible regime with any compressible one. It is shown that quite a general static equilibrium is not less stable than the same basic state of compressible fluid.

Introduction

Some of the results of a previous paper [1] on the relations between adiabatic and incompressible systems are the establishing of a variational energy principle for incompressible systems and consequently of comparison theorems for stability of adiabatic and incompressible perturbations of the same equilibrium state. In this paper we extend these analyses to the most general compressible regimes which are consistent with the variational technique (Hermiticity condition). Moreover, we extend the treatment to the case of anisotropic pressure in the last section.

For a uniform fluid which is adiabatically disturbed we have for the square of the oscillation frequency

$$\omega^2 = c^2 \int (\operatorname{div} \boldsymbol{\xi})^2 \, d\tau / \int \boldsymbol{\xi}^2 \, d\tau \qquad (1)$$

where c is the speed of sound. The minimum ω^2 is attained for $\operatorname{div} \boldsymbol{\xi} = 0$. For the linear diffuse pinch in magnetodynamics, $\operatorname{div} \boldsymbol{\xi}$ is zero at marginal stability. This was first pointed out by Shafranov [2]. The deeper ground of this fact was given in our previous paper [1] and the class of problems for which $\operatorname{div} \boldsymbol{\xi} = 0$ at marginal stability was determined. For this class of problems the compressible and adiabatic regimes are both stable or both unstable.

In view of these examples and the mathematical simplification obtained in putting $\operatorname{div} \boldsymbol{\xi} = 0$, one has usually considered the incompressibility approximation whe calculating growth rates, expecting an incompressible displacement to provide the minimum ω^2. This can only be true approximately. Indeed, we prove in Section 3 that quite a general state of static equilibrium of incompressible fluid is more stable or at least equally stable than the same basic state of compressible fluid. We point out that this statement may not be confused with the (much more restricted) comparison of the incompressible modes of the compressible regime and its compressible modes, for which the statement is of course also valid.

1. Compressible Regime

Consider a compressible non-dissipative medium with, for the time being, an isotropic pressure tensor. The equation of motion is

$$\varrho \frac{d\boldsymbol{v}}{dt} = -\operatorname{grad} p + \boldsymbol{f} \qquad (2)$$

where ϱ indicates as usual the mass density, $\boldsymbol{v}$ the velocity, p the pressure and where $\boldsymbol{f}$ is the force per unit volume. The perturbed equation of motion is

$$\varrho \ddot{\boldsymbol{\xi}} = \boldsymbol{F}(\boldsymbol{\xi}) \qquad (3)$$

$$= -\operatorname{grad} \delta p + \delta \boldsymbol{f} \qquad (4)$$

where $\boldsymbol{\xi}$ is the infinitesimal displacement.

We make the following two suppositions:

a) F is an Hermitian operator

$$\int \boldsymbol{\eta} \cdot \boldsymbol{F}(\boldsymbol{\xi}) \, d\tau = \int \boldsymbol{\xi} \cdot \boldsymbol{F}(\boldsymbol{\eta}) \, d\tau \, . \qquad (5)$$

For simplicity we assume that the integration is carried out over the whole fluid extending to infinity.

b) The perturbed force δf depends upon neither the pressure nor the heat flow

Supposition a) implies that the right hand side

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[1] J. G. Krüger and D. K. Callebaut, Z. Naturforsch. **25 a**, 88 [1970].

[2] V. D. Shafranov, Physics of Plasmas and Problems of Controlled Thermonuclear Reactions, Pergamon Press, New York 1960, Vol. 4, p. 71.

of Eq. (3) depends on $\boldsymbol{\xi}$ but not on $\dot{\boldsymbol{\xi}}$ and that the eigenvalues $-\omega^2$ of $\varrho^{-1}\boldsymbol{F}$ are real.

Clearly these two suppositions are not very restrictive. They are satisfied e. g. in magnetodynamics [3], gravitodynamics [4] and magnetogravitodynamics [5]. In all these theories the variation of the pressure (following the motion) is the adiabatic law:

$$\mathrm{d}p = -\gamma\, p \operatorname{div} \boldsymbol{\xi}. \tag{6}$$

We shall discuss here the general law of pressure variation

$$\mathrm{d}p = \alpha(\boldsymbol{\xi}), \tag{7}$$

$\alpha(\boldsymbol{\xi})$ for the time being an arbitrary function of $\boldsymbol{\xi}$. On account of supposition b) it is possible to separate the force $\boldsymbol{F}(\boldsymbol{\xi})$ in two terms

$$\boldsymbol{F}(\boldsymbol{\xi}) = \boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi}) + \boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi}) \tag{8}$$

where

$$\boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi}) = -\operatorname{grad} \mathrm{d}p \tag{9}$$

is entirely dependent on the equation of state (7) and $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ is totally independent of it.

In all the mentioned fluid approaches based on the adiabatic law (6) we have according to Gauss' theorem

$$\int \boldsymbol{\eta} \cdot \boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})\, \mathrm{d}\tau = \int \gamma\, p \operatorname{div} \boldsymbol{\xi} \operatorname{div} \boldsymbol{\eta}\, \mathrm{d}\tau \tag{10}$$

from which it is clear that $\boldsymbol{F}_{\mathrm{d}p}$ is Hermitian. As $\boldsymbol{F}$ and $\boldsymbol{F}_{\mathrm{d}p}$ are Hermitian, $\boldsymbol{F}_{\mathrm{d}p=0}$ is Hermitian. Extending the adiabatic law to a more general compressible law the term $\boldsymbol{F}_{\mathrm{d}p=0}$ remains unaltered if $\delta\boldsymbol{f}$ is independent of pressure and heat flow as required by our supposition b). Concluding, as the operator F is assumed to be Hermitian for any law of pressure variation, each of the operators $F_{\mathrm{d}p}$ and $F_{\mathrm{d}p=0}$ are necessarily Hermitian separately.

The Hermiticity of $\boldsymbol{F}_{\mathrm{d}p}$ restricts the possible expressions for $\alpha(\boldsymbol{\xi})$. Indeed, using Gauss' theorem and Eq. (7) we have

$$\int \boldsymbol{\eta} \cdot \boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})\, \mathrm{d}\tau = \int \alpha(\boldsymbol{\xi}) \operatorname{div} \boldsymbol{\eta}\, \mathrm{d}\tau. \tag{11}$$

Hermiticity of $\boldsymbol{F}_{\mathrm{d}p}$ requires

$$\int \alpha(\boldsymbol{\xi}) \operatorname{div} \boldsymbol{\eta}\, \mathrm{d}\tau = \int \alpha(\boldsymbol{\eta}) \operatorname{div} \boldsymbol{\xi}\, \mathrm{d}\tau. \tag{12}$$

This equation can be only satisfied if

$$\alpha(\boldsymbol{\xi}) = \mathcal{H} \operatorname{div} \boldsymbol{\xi} \tag{13}$$

where $\mathcal{H}$ is a real but otherwise arbitrary Hermitian operator. If only first and second order derivatives should occur in $\mathcal{H}$, it is necessarily of the form

$$\mathcal{H} = -\Gamma\, p + \sum_{\alpha,\beta} \frac{\partial}{\partial x_\alpha} \varphi_{\alpha\beta} \frac{\partial}{\partial x_\beta} \tag{14}$$

where Γ and $\varphi_{\alpha\beta}$ are arbitrary functions of position. The total variation of the pressure is then

$$\mathrm{d}p = -\Gamma\, p \operatorname{div} \boldsymbol{\xi} + \sum_{\alpha,\beta} \frac{\partial}{\partial x_\alpha} \varphi_{\alpha\beta} \frac{\partial \operatorname{div} \boldsymbol{\xi}}{\partial x_\beta}. \tag{15}$$

This equation has to be compared with the perturbed form of the general equation for the variation of thermal energy

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\gamma\, p \operatorname{div} \boldsymbol{v} + (1-\gamma) \operatorname{div} \boldsymbol{q} \tag{16}$$

where $\boldsymbol{q}$ stands for the heat flow. A possible identification from comparing Eqs. (15) and (16) is $\Gamma = \gamma = \text{constant}$ and

$$\boldsymbol{q} = \frac{\varphi}{1-\gamma} \operatorname{grad} \operatorname{div} \boldsymbol{v} \tag{17}$$

with the assumption that $\varphi_{\alpha\beta}$ is isotropic ($\varphi_{\alpha\beta} = \varphi\, \delta_{\alpha\beta}$). Eq. (17) is different from Fourier's law $\boldsymbol{q} \sim \operatorname{grad} T$ to first order (T = temperature), which gives always rise to an operator $\boldsymbol{F}$ which is not Hermitian, this being the mathematical expression of a damping effect. By supposition of Hermiticity all damping processes are excluded from our description beforehand.

As a consequence of the Hermitivity of $\boldsymbol{F}$ the equation of motion can be derived from the variation of the action with respect to the displacement $\boldsymbol{\eta}$. The Lagrangian is

$$\mathcal{L} = \delta K(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}) - \delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) \tag{18}$$

where

$$\delta K(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}) = \tfrac{1}{2} \int \varrho\, \dot{\boldsymbol{\xi}}^2\, \mathrm{d}\tau \tag{19}$$

represents the kinetic energy involved in the perturbation and where the energy integral, $\delta W(\boldsymbol{\xi}, \boldsymbol{\xi})$, is composed of two terms according to Eq. (8)

$$\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) = \delta W_{\mathrm{d}p}(\boldsymbol{\xi}, \boldsymbol{\xi}) + \delta W_{\mathrm{d}p=0}(\boldsymbol{\xi}, \boldsymbol{\xi}) \tag{20}$$

the first term being entirely dependent on $\alpha(\boldsymbol{\xi})$, the second one being independent of it. We have

$$\delta W_{\mathrm{d}p=0}(\boldsymbol{\xi}, \boldsymbol{\xi}) = -\tfrac{1}{2} \int \boldsymbol{\xi} \cdot \boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})\, \mathrm{d}\tau, \tag{21}$$

$$\delta W_{\mathrm{d}p}(\boldsymbol{\xi}, \boldsymbol{\xi}) = -\tfrac{1}{2} \int \alpha(\boldsymbol{\xi}) \operatorname{div} \boldsymbol{\xi}\, \mathrm{d}\tau \tag{22}$$

using Gauss' theorem.

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2. Incompressible Regime

Compare now any compressible regime having a general law of pressure variation (7) with the incompressible regime. We suppose again that the two suppositions a) and b) of Section 1 are valid. According to supposition b) the part $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ remains the same as in the compressible regime. Eq. (7) is now replaced by

$$\operatorname{div} \boldsymbol{\xi} = 0 . \tag{23}$$

Hence the force term $\boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})$ has to stay in the form (9) where dp has to be considered now as a fourth variable. The vector equation (3) together with Eq. (23) make indeed just four equations in the four unknowns $\boldsymbol{\xi}$ and dp. We state now the extensions of theorem 1 of our previous paper [1] (which deals only with incompressible and adiabatic perturbations).

Variational Principle for the Incompressible Regime

a) *In order to obtain the energy integral for incompressible perturbations of a static equilibrium it is sufficient to put* $\operatorname{div} \boldsymbol{\xi} = 0$ *in the energy integral for compressible perturbations;*

b) *The energy principle technique is valid for incompressible perturbations if the displacements* ξ *are restricted to solenoidal ones* ($\operatorname{div} \boldsymbol{\xi} = 0$).

Statement a) is proved by the remark that

$$\delta W_{\mathrm{d}p}(\boldsymbol{\xi},\boldsymbol{\xi}) = 0$$

for the incompressible regime [see Eq. (22)]. To prove statement b) we follow the Lagrange multiplier technique and use a Lagrange multiplier in every point of the fluid. We add the term

$$\int a \operatorname{div} \boldsymbol{\xi} \mathrm{d}\tau = -\int \boldsymbol{\xi} \cdot \operatorname{grad} a \, \mathrm{d}\tau + \int a \, \boldsymbol{n} \cdot \boldsymbol{\xi} \mathrm{d}\sigma \tag{24}$$

to the Lagrangian (18). The consequence is the introduction of a supplementary term $-\operatorname{grad} a$ in the equation of motion. In this aspect this term replaces $\boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})$ and the Lagrangian multiplier a replaces the function $\alpha(\xi)$ in the equation of motion. The variation of $\delta W_{\mathrm{d}p=0}(\boldsymbol{\xi},\boldsymbol{\xi})$ gives $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ and thus we obtain

$$\varrho \ddot{\boldsymbol{\xi}} = -\operatorname{grad} a + \boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi}) . \tag{25}$$

The system of Eqs. (22) and (23) constitutes just four equations in the four unknowns $\boldsymbol{\xi}$ and a. These equations are therefore identical with those for the incompressible regime if we identify a with dp. Other identifications of a are possible if, as in the following section, we should put $\operatorname{div} \boldsymbol{\xi} = 0$ in the energy integral after the variation instead of before it.

3. Comparison of Stability

We now compare the stability of a given equilibrium state, governed first by the incompressible law (23) and secondly by the general Equation (7). In order to do this we seek normal mode solutions of the form $\boldsymbol{\xi}(\boldsymbol{r}, t) = e^{i\omega t} \boldsymbol{\xi}(\boldsymbol{r})$. The eigenvalues ω and the eigenmodes $\boldsymbol{\xi}_\omega$ are given by the variational problem $\Delta\omega^2 = 0$ where

$$\omega^2(\boldsymbol{\xi}, \boldsymbol{\xi}) = \delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) / K(\boldsymbol{\xi}, \boldsymbol{\xi}) . \tag{26}$$

As is well known ω^2 is always real because $\boldsymbol{F}$ is Hermitian and $\omega^2 < 0$ corresponds to the unstable modes while $\omega^2 > 0$ corresponds to the oscillatory ones. As in Sections 2 and 3 we have to vary $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$ without constraint in the compressible regime and with the constraint (23) in the incompressible regime. We remark however that we may use *exactly* the same functional $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$ of the compressible regime in both problems. Indeed, variation of $\omega^2(\boldsymbol{\xi}, \boldsymbol{\xi})$ following the Lagrange multiplier technique of Section 2 yields

$$-\omega^2 \boldsymbol{\xi} = -\operatorname{grad}[\alpha(\boldsymbol{\xi}) + a] + \boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi}) . \tag{27}$$

Eqs. (23) and (27) are just four equations in the four unknowns $\boldsymbol{\xi}$ and $\alpha(\boldsymbol{\xi}) + a$. They are identical with the equations for the incompressible case if we identify dp now with $\alpha(\boldsymbol{\xi}) + a$. Hence the lowest eigenvalue for the compressible regime is given by the absolute minimum of $\omega^2(\boldsymbol{\xi}, \boldsymbol{\xi})$ say $\omega_0{}^2$. The lowest eigenvalue $\omega_{0,\mathrm{I}}^2$ for the incompressible regime satisfies necessarily

$$\omega_0^2 \leqq \omega_{0,\mathrm{I}}^2 \tag{28}$$

because it is obtained from a constrained variation. From Eq. (28) we see that, if $\omega_{0,\mathrm{I}}^2$ is negative the fastest growing incompressible perturbation will never grow faster than any compressible one. We conclude to the following

Comparison Theorem

A static equilibrium state is more stable or at least equally stable for incompressible perturbations than for any compressible ones for a physical system for which the general suppositions (a) and (b) are satisfied.

Hence, if an equilibrium state proves to be unstable for incompressible perturbations (for which stability is easier to study) then it is also unstable for all possible compressible regimes (but not conversely).

If the compressible regime is in particular the adiabatic one characterized by Eq. (6), then the theorem reduces to theorem 2 of our previous paper [1]. The proof given here is a more direct one, which has no recourse to the lemma of the paper cited above. From the derivation of the theorem it is clear that it is also valid for the comparison of the incompressible modes of the compressible regime with its compressible modes.

The occurrence of the constraint (23) in the incompressible regime and its absence in the compressible regime has its physical origin in the fact that Eq. (23) implies, in one way or another, a stronger restriction on the behaviour of the fluid than Eq. (7) does. Indeed, Eq. (23) is a constraint which restricts the *spatial* distribution of the velocities *at any instant.* This is a severe restriction: it does not allow, in particular, that the initial displacement can be chosen freely. On the other hand, Eq. (7) determines only the *evolution* of a physical quantity: it expresses a relation between two infinitesimal close instants. This is a less severe restriction; in particular it allows physically to suppose that the initial displacement can now be chosen freely.

For a normal mode, if there exists a constraint on the spatial distribution of the displacement vector at the initial time, this constraint evidently exists at any time. We conclude that the occurrence of a *constraint on the initial conditions* in the incompressible regime explains physically why this regime is the most stable one, if there are no other differences in the behaviour of the system. This restriction indicates that the form of the energy integral has to be exactly the same in both regimes in order that the theorem is valid. The simple results of the comparison theorem and of the variational principle of Section 2, are also due to the fact that, in the incompressible regime, no supplementary term due to the heat flow occurs in δW, as compared with the compressible regime.

It is well known that for the Kelvin-Helmholtz instability, the incompressible regime is less stable than the adiabatic one. From this example we infer that the theorem cannot be extended to all states of steady motion with full generality. This shows that, although the intuitive content of the comparison theorem may now seem simple, the conditions and limits of application cannot easily be understood by intuition.

4. Extension to Anisotropic Pressure

The analyses can easily be extended to include anisotropic pressure if we define carefully what is meant by an incompressible regime in this case. For convenience we write the pressure tensor in the form

$$p_{ij} = p\,\delta_{ij} + \mathring{p}_{ij} \tag{29}$$

where p is the mean hydrostatic pressure, δ_{ij} is the Kronecker delta and $\mathring{p}_{ij}$ is the traceless part of the pressure tensor. The equation for the variation of p is still given by Eq. (16); the variations of $\mathring{p}_{ij}$ are irrelevant in our discussion. For the incompressible regime the condition (23) must replace one of the equations giving the variations of p or $\mathring{p}_{ij}$. We choose that Eq. (23) replaces Eq. (16). In accordance with this choice we suppose also that $\mathring{p}_{ij}$ remains unchanged in going over from the compressible to the incompressible regime.

This supposition is very natural for e. g. the description of viscosity. Indeed, the effect of viscosity is described by a traceless pressure tensor proportional with the (traceless) strain of the fluid, while the mean hydrostatic pressure p is still determined by the equation of thermal energy. However the simple example of viscosity lies outside the scope of this article because it leads to a damping effect which is excluded by our Hermiticity assumption.

In the equation of motion (3) we again may decompose the force $\boldsymbol{F}(\boldsymbol{\xi})$ into two components $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ and $\boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})$; $\boldsymbol{F}_{\mathrm{d}p}(\boldsymbol{\xi})$ is entirely dependent on the main hydrostatic pressure and $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ is totally independent of it. $\boldsymbol{F}_{\mathrm{d}p=0}(\boldsymbol{\xi})$ contains now the force term of Sections 2 and 3 plus the part coming from the traceless pressure $\mathring{p}_{ij}$. All the arguments of Sections 2 and 3 can be repeated for this case. The variational principles and the comparison theorem are therefore valid for the case of anisotropic pressure if the two regimes differ only by the variation of the mean hydrostatic pressure.

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